International Journal of Theoretical Physics, Vol. 23, No. 10, 1984

Charge–Magnetic-Dipole Bound States Using the Klein–Gordon Equation

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Received November 17, 1983

We investigate bound states of a composite system consisting of a charged particle orbiting a neutral, stationary magnetic dipole. We find all bound states are metastable and none exist with angular momentum less than eleven. Our calculation is performed in two space dimensions.

1. INTRODUCTION

Composite models of quarks and leptons are currently a popular topic of theoretical investigation although there is no direct experimental evidence for any internal structure. Two of the prime reasons for this popularity are the proliferation of quarks and leptons and their apparent grouping into families. If quarks and leptons are assumed to be elementary, the first fact creates a contradiction with the esthetically pleasing idea that all matter is constructed using a small number of elementary building blocks. The second fact can be readily understood if an underlying structure exists. An explanation for the existence and properties of quark–lepton families would then parallel the historical explanations of the periodicity of the elements using an atomic model and the grouping of hadrons into families using a quark model.

Most approaches to composite models concentrate on the properties required by the constituents so that the known families of the quarks and leptons can be constructed; see, for example, Greenberg and Nelson (1974), Bars and Gunaydin (1980), Derman (1980), Harari (1979), Barbieri, Maiani, and Petronzio (1980), DeRujula (1980), and Terazawa and Akama (1980).

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We report here on progress of an investigation which is similar in spirit to those by Kopper and Durr (1981) and (1982), Greenberg and Sucher (1981), Bander, Chiu, Shaw, and Silverman (1982), and Tomozawa (1982). We examine a specific dynamical structure, namely, magnetic binding of two constituents—one a magnetic dipole and the other an electric charge. The ultimate hope of this type of approach is that it will be possible to explain why all quarks and leptons have spin-1/2, to understand the arrangement of quarks and leptons into families, and to calculate—among other dynamical properties—the decay rates of the heavier quarks and leptons. Here we restrict our attention to the properties of leptons since their lack of color makes them simpler than quarks.

It must be considered somewhat heretical to think electromagnetism could be the interaction which binds together constituents within an electron. On the other hand, the electron–muon mass ratio being a simple multiple of the fine structure constant, $m_e/m_\mu = 2\alpha/3$, certainly suggests an electromagnetic origin for differentiating between lepton families. Furthermore, Barut and Kraus (1976) have shown that under certain circumstances, systems bound with a magnetic force can have a radius comparable to or smaller than the experimental upper limit on the radius of the electron, so the magnetic force is capable of the strong binding required.

Our ideas are not new, but their specific application to composite lepton models has not been made in this way before. Barut and coworkers (Barut, 1981, 1980, 1979) have considered magnetically bound composite models of leptons but have used known particles as the constituents. Much earlier, Schild (1963) studied the extreme relativistic motion of a point charge in a magnetic dipole field both classically and using Bohr–Sommerfeld quantization. Schild's work is similar in spirit to ours in that we have chosen not to identify the constituents with known particles. By specifying the properties of these constituents, we push back to another level of understanding the nature of the constituents.

In earlier work (Mainland and Scott, 1981, 1982) we considered, within the framework of a two-dimensional, nonrelativistic calculation, a bound state consisting of a charged particle and a charged magnetic dipole. From numerical calculations, we found a large gap in the allowed values of orbital angular momentum of the system for a certain range of the strength of the dipole. The s state was always allowed while the orbital angular momenta l for other states were restricted to the range $l \geqslant 11$ for a sufficiently small magnetic dipole moment. The states with zero orbital angular momentum are the most tightly bound so if this mechanism were operable in a realistic composite model of leptons, it could explain why the observed leptons all have spin-1/2.

2. THE RELATIVISTIC, CHARGE–NEUTRAL-DIPOLE SYSTEM

In this work, we consider a spin-0 particle with a mass m and a charge q orbiting an electrically neutral magnetic dipole moment fixed at the origin. We have employed the Klein–Gordon equation for the orbiting particle in order to include relativistic effects. Since there is no Coulomb interaction, the calculation is greatly simplified. In addition, since the Coulomb interaction is responsible for all $l=0$ bound states, they will be absent here. We find that in this model there is a finite, lower limit on the allowed values of orbital angular momenta: $l \geqslant 11$. Work is in progress on an extension of this model which includes a Coulomb interaction. Because the magnetic potential resulting from a magnetic dipole is cylindrically—but not spherically—symmetric, we are unable to separate the Klein–Gordon equation in three dimensions but can do so in two dimensions, so we restrict our attention to the latter case. In cylindrical coordinates the magnetic potential[1] resulting from a dipole is [see Jackson (1962)] $\mathbf{A}(r) = (\mu\mu_0/4\pi r^2)\hat{\phi}$. Making a minimal substitution in the time-independent Klein–Gordon equation yields

$$E^2\psi = \left[c^2\left(-i\hbar\frac{\partial}{\partial x} - qA_x\right)^2 + c^2\left(-i\hbar\frac{\partial}{\partial y} - qA_y\right)^2 + m^2c^4\right]\psi \qquad (1)$$

where $A_x = -(\mu\mu_0/4\pi r^2)\sin\phi$, $A_y = (\mu\mu_0/4\pi r^2)\cos\phi$, and $\tan\phi = y/x$. Rewriting the equation in cylindrical coordinates

$$E^2\psi = \left[-\hbar^2c^2\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + c^2\left(-i\frac{\hbar}{r}\frac{\partial}{\partial\phi} - \frac{q\mu\mu_0}{4\pi r^2}\right)^2\psi \qquad (2)$$

Because the magnetic field of the dipole is cylindrically symmetric, the Hamiltonian commutes with the z component of angular momentum and the wave function ψ can be written as a simultaneous eigenstate of both operators:

$$\psi = e^{il\phi}\chi(r) \qquad (3)$$

The radial equation for χ is then immediately found to be

$$\frac{d^2\chi}{dr^2} + \frac{1}{r}\frac{d\chi}{dr} - \left(\frac{l}{r} - \frac{q\mu\mu_0}{4\pi\hbar r^2}\right)^2\chi + \left(\frac{E^2 - m^2c^4}{\hbar^2c^2}\right)\chi = 0 \qquad (4)$$

[1] We use Systeme Internationale units in this paper.

To solve this equation we use the WKB approximation and make the independent variable transformation $r = \exp(z)$ to put equation (4) into standard form:

$$\frac{d^2\chi}{dz^2} + \frac{e^{2z}}{\hbar^2}\left[\frac{E^2 - m^2c^4}{c^2} - \left(l\hbar e^{-z} - \frac{q\mu\mu_0}{4\pi}e^{-2z}\right)^2\right]\chi = 0 \tag{5}$$

The WKB quantization condition [see Park (1974) and Morse and Feshbach (1963)] is immediately seen to be

$$(n + 1/2)\pi\hbar = \int_{z_1}^{z_2} e^{z}\left[\varepsilon^2 - \left(l\hbar e^{-z} - \beta e^{-2z}\right)^2\right]^{1/2} dz \tag{6}$$

where $\varepsilon^2 = (E^2 - m^2c^4)/c^2$ and $\beta = q\mu\mu_0/4\pi$. The turning points z_1 and z_2 are solutions of the equation

$$0 = \varepsilon^2 - \left(l\hbar e^{-z} - \beta e^{-2z}\right)^2 \tag{7}$$

Listed in order of increasing size, the three positive turning points are readily found to be

$$e^{z_1} \equiv r_1 = \frac{\left(l^2\hbar^2 + 4\varepsilon\beta\right)^{1/2} - l\hbar}{2\varepsilon}$$

$$e^{z_2} \equiv r_2 = \frac{-\left(l^2\hbar^2 - 4\varepsilon\beta\right)^{1/2} + l\hbar}{2\varepsilon}$$

$$e^{z_3} \equiv r_3 = \frac{\left(l^2\hbar^2 - 4\varepsilon\beta\right)^{1/2} + l\hbar}{2\varepsilon} \tag{8}$$

The term $(l\hbar e^{-z} - \beta e^{-2z})^2 = (l\hbar/r - \beta/r^2)^2$ acts as an effective potential V_{eff} and is plotted as a function of r in Figure 1. Values of $l = 4$ and $l = 8$ have been used to show how the shape of the potential changes with orbital angular momentum. Note that the well in which the particle of mass m and charge q is bound becomes deeper and narrower as l increases. Since $V_{\text{eff}} \geqslant 0$ and approaches zero as r goes to infinity, all bound states are metastable. For small values of orbital angular momentum, the effective potential well is not sufficiently deep to create a metastable state. For successively larger values of l, however, the well changes shape and its area increases until, at some value of l, a metastable state is created. Such metastable bound states occur only for ε^2 less than the maximum height of

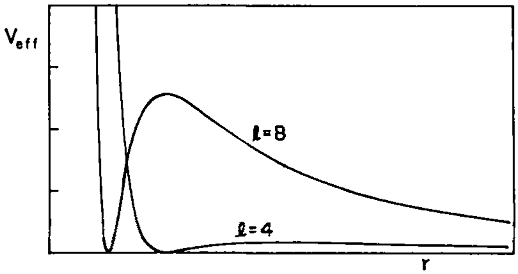

Fig. 1. Effective potential for the charge-magnetic dipole system.

the barrier or for

$$\varepsilon < \frac{l^2\hbar^2}{4\beta} \tag{9}$$

which is identical to the condition that there be three real, positive turning points.

The phase integral in (6) is evaluated by changing to the independent variable $x = \sqrt{\beta}\, e^{-z} - l\hbar/2\sqrt{\beta}$. In terms of x the quantization condition (6) becomes

$$(n+1/2)\pi\hbar = -\sqrt{\beta}\int_{x_1}^{x_2}\left[\varepsilon^2 - \left(x^2 - \frac{l^2\hbar^2}{4\beta}\right)^2\right]^{1/2}\left(x + \frac{l\hbar}{2\sqrt{\beta}}\right)^{-2} dx \tag{10}$$

where $x_1 = [l^2\hbar^2/4\beta - \varepsilon]^{1/2}$ and $x_2 = [l^2\hbar^2/4\beta + \varepsilon]^{1/2}$. Integrating by parts and using the fact $\varepsilon^2 - (x^2 - l^2\hbar^2/4\beta)^2$ vanishes at the turning points x_1 and x_2, (10) takes the form

$$\left(n + \frac{1}{2}\right)\pi\hbar = -\sqrt{\beta}\int_{x_1}^{x_2}\frac{\left[2x^2 - (l\hbar/\sqrt{\beta})x\right]dx}{\left[(x_2^2 - x^2)(x^2 - x_1^2)\right]^{1/2}} \tag{11}$$

The integral can be readily evaluated (Dwight, 1966) with the result that the quantization condition becomes

$$\left(n + \frac{1}{2}\right)\pi\hbar = -2\sqrt{\beta}\left(\frac{l^2\hbar^2}{4\beta} + \varepsilon\right)^{1/2} E(k) + \frac{\pi}{2} l\hbar \tag{12}$$

where $E(k)$ is the complete elliptic integral of the second kind and

$$k = \left[\frac{2\varepsilon}{(l^2\hbar^2/4\beta)+\varepsilon} \right]^{1/2} \tag{13}$$

We note that $k \leqslant 1$ and examine two extreme cases. The first occurs when $\varepsilon = l^2\hbar^2/4\beta$ (the maximum height of the potential barrier). Lower energy states actually occur for this case because lower values of the orbital angular momentum are allowed. From (13), we see that the above condition on ε implies $k = 1$. Since the phase integral is a maximum under these conditions, it follows from the quantization that

$$\left(n+\frac{1}{2}\right)\pi\hbar \leqslant \left[-2\sqrt{\beta}\left(\frac{l^2\hbar^2}{4\beta}+\varepsilon\right)^{1/2} E(k=1) + \frac{\pi}{2} l\hbar \right]\Bigg|_{\varepsilon = l^2\hbar^2/4\beta} \tag{14}$$

Using [see Dwight (1966)] $E(k) = 1$, (14) yields the constraint

$$l \geqslant \frac{(2n+1)\pi}{\pi - 2\sqrt{2}} \tag{15}$$

Taking $n = 0$, the minimum value of orbital angular momentum for metastable states is found to be 11.

The other extreme case corresponds to metastable states with energies substantially below the relative maximum of the effective potential which occur for $l > 11$ and correspond to $\varepsilon \ll l^2\hbar^2/4\beta$ or $k \ll 1$. $E(k)$ can now be expanded in a power series [see Dwight (1966)] in k,

$$E(k) \simeq \frac{\pi}{2}\left(1 - \frac{2\varepsilon\beta}{l^2\hbar^2} + \frac{5\varepsilon^2\beta^2}{l^4\hbar^4} + \cdots \right), \qquad k \ll 1 \tag{16}$$

The quantization condition then becomes

$$\left(n+\frac{1}{2}\right)\pi\hbar \simeq \frac{\pi}{2}\frac{\varepsilon^2\beta^2}{l^3\hbar^3} \tag{17}$$

or, in terms of the original variables,

$$E^2 \simeq m^2c^4 + \frac{16\pi^2c^2\hbar^4(2n+1)l^3}{q^2\mu^2\mu_0^2} \tag{18}$$

By using the approximate relation (16) in the constraint (9) we arrive at the condition $l > 16(2n+1)$. Since the minimum value of n is zero, this yields a minimum value of 17 for the orbital angular momentum. Within the context of the WKB approximation, however, we found the exact result for this quantity to be 11. The difference between these values simply reflects the error made in using the power series expansion for $E(k)$ in (16) when $k=1$.

When we included a Coulomb interaction in a nonrelativistic version of this model (Mainland and Scott 1981), the result of a numerical calculation was that the minimum nonzero allowed value of orbital angular momentum was also 11.

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